i) Write down Little's theorem and define all terms. (10%)

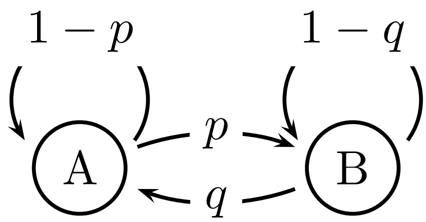
- $N = \lambda T$. (5%)
- N is mean number in queue (5%)
- λ is mean arrival rate to queue (5%)
- T is mean time spent in queue (5%)

ii) Give the conditions under which it holds. (10%)

Let α be the number of arrivals to time t. Let β be the number of departures to time t. Let T_t be the mean waiting time up to time t. The four conditions earn (2.5%) each – 5% docked for adding unnecessary conditions:

1. The limit $\lambda = \lim_{t\to\infty} \alpha(t)/t$ exists
2. The limit $\delta = \lim_{t\to\infty} \beta(t)/t$ exists
3. The limit $T = \lim_{t\to\infty} T_t$ exists
4. $\delta = \lambda$

iii) Consider a two state discrete time Markov chain with states A and B. The transition probability from A to B is p (where $0 < p < 1$) and from B to A is q (where $0 < q < 1$). There are also transitions from A and B to themselves. Draw the chain and all transitions. (10%).



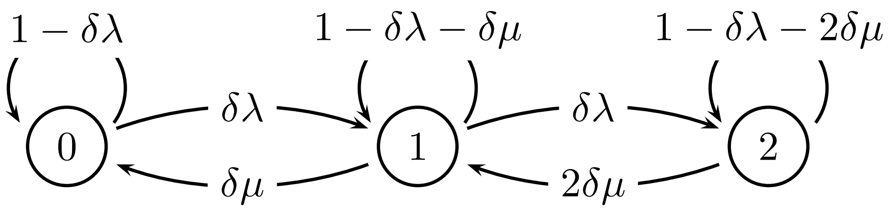
iv) Calculate the equilibrium probabilities π_A and π_B for the chain in iii) in terms of p and q. (20%)

$$
\begin{aligned}
\pi_A + \pi_B &= 1 && \text{probabilities sum to 1} \\
\pi_A &= q\pi_B + (1-p)\pi_A && \text{balance eqn state A} \\
p\pi_A &= q\pi_B && \text{rearrange above} \\
p\pi_A &= q(1-\pi_A) && \text{substitute first line} \\
\pi_A &= q/(p+q) && \text{rearrange} \\
\pi_B &= p/(p+q) && \text{from line 1}
\end{aligned}
$$

v) Consider an M/M/2 queue. What do the first and second M's mean? What does the 2 mean? (10%)

The arrival rate is "memoryless" (Poisson). The server departure rate is "memoryless" (Poisson). There are two servers.

vi) In an M/M/2/n queue the final n is the maximum number in the queue. This queue will be approximated with a discrete time Markov chain. The arrival rate is λ and the small time step is δ so that $\lambda\delta$ is the mean number of arrivals in one time step. The rate of a single server is μ. Draw a discrete time Markov chain which approximates the M/M/2/2 queue where the number of the state is the number in the queue. Include all transition probabilities. You may assume δ is so small that all relevant transition probabilities are in the range (0,1). (20%)



vii) Calculate the equilibrium probability π_0 for the chain from vi) in terms of λ and μ (you may use without derivation any formulas taught in class). (20%)

Direct derivation:

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad \text{probabilities sum to 1}$$
$$\pi_0 = (1-\delta\lambda)\pi_0 + \delta\mu\pi_1 \qquad \text{balance eqn state 0}$$
$$\pi_1 = \pi_0\lambda/\mu \qquad \text{rearrange}$$
$$\pi_1 = \delta\lambda\pi_0 + (1-\delta\lambda-\delta\mu)\pi_1 + 2\delta\mu\pi_2 \qquad \text{balance state 1}$$
$$\pi_2 = \pi_0\lambda^2/2\mu_2 \qquad \text{rearrange}$$
$$1 = \pi_0 + \pi_0\lambda/\mu + \pi_0\lambda^2/2\mu^2 \qquad \text{from line 1}$$
$$\pi_0 = 1/(1+\lambda/\mu+\lambda^2/2\mu^2)$$

Quick derivation via general birth death process gives this answer from:

$$\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i}}.$$

Fiven $\lambda_i = \lambda$ for all i and $\mu_1 = \mu$ and $\mu_2 = 2\mu$ this becomes

$$\pi_0 = 1/(1+\lambda/\mu+\lambda^2/2\mu^2).$$