Lecture 10 — Networks of queues

In this lecture we shall finally get around to consider what happens when queues are part of networks (which, after all, is the topic of the course). Firstly we shall need an important result about time reversibility and Markov chains.

Burke's Theorem

Consider an ergodic Markov chain $\{X_j : j \in \mathbb{N}\}$ which is Markov $(\lambda, \mathbf{P})$ running backwards (or if you prefer consider the reverse iterates of the Markov chain). This asks questions about

$$\begin{aligned}
&\mathbb{P}\left[X_n = j | X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k\right] \\
=&\frac{\mathbb{P}\left[X_n = j, X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k\right]}{\mathbb{P}\left[X_{n+1} = i, X_{j+2} = i_2, \ldots, X_{n+k} = i_k\right]} \\
=&\frac{\mathbb{P}\left[X_n = j, X_{n+1} = i\right] \mathbb{P}\left[X_{n+2} = i_2, \ldots, X_{n+k} = i_k | X_n = j, X_{n+1} = i\right]}{\mathbb{P}\left[X_{n+1} = i\right] \mathbb{P}\left[X_{n+2} = i_2, \ldots, X_{n+k} = i_k | X_{n+1} = i\right]} \\
=&\frac{\mathbb{P}\left[X_n = j, X_{n+1} = i\right]}{\mathbb{P}\left[X_{n+1} = i\right]} \\
=&\frac{\mathbb{P}\left[X_n = j\right] \mathbb{P}\left[X_{n+1} = i | X_n = j\right]}{\mathbb{P}\left[X_{n+1} = i\right]} \\
=&\frac{\pi_j p_{ji}}{\pi_i},
\end{aligned}$$

The fourth line follows from the Markov property and for the last equality the assumption has been made that the chain has reached its steady-state equilibrium probabilities.

Denote by p^*_{ij} the reversed transition probability

$$p^*_{ij} = \mathbb{P}\left[X_n = j | X_{n+1} = i\right] = \frac{\pi_j p_{ji}}{\pi_i}. \tag{1}$$

The reversed chain is ergodic and has the same equilibrium probabilities (check that $\pi_j = \sum_{i=0}^{\infty} \pi_i p^*_{ij}$). Intuitively think of the film of the Markov chain in action being shown backwards.

A chain is called *time reversible* if $p_{ij} = p^*_{ij}$ for all i and j. Clearly, from equation (1) this occurs iff

$$p_{ij}\pi_i = p_{ji}\pi_j,$$

for all i, j.

Note that this means that Birth-Death processes are time reversible (the proof of this is left as an exercise for the student). Therefore all the queues which can be modelled as such are, themselves time reversible. This includes the M/M/1, the M/M/m and the M/M/∞ queues. Therefore, queue which can be represented as a Birth-Death process can be considered, once it has reached the steady state. An important point here is that the departure process of the forward system is the arrival process of the forward system.

This leads us to Burke's Theorem.

Theorem 1. *For an M/M/1, M/M/m or M/M/∞ queue in the steady state then: (1) The departure process is Poisson with rate λ. (2) At time t the number of customers is independent of the sequence of departure times prior to t.*

Proof. Part (1) follows immediately from the fact that the arrival process is Poisson with rate λ the time reverse of this of this is also a Poisson with rate λ.

Part (2) follows from the fact that the departures prior to t in the reversed system is the same process as the arrival process after t in the reversed process. It is clear that the number in the system queue is independent of the arrivals after that point in a Poisson system. □

Note that these results are more than a little counter-intuitive. In (1) The exit rate is not a function of the server rate. In (2) a sequence of closely spaced exits does not mean that the system is at all likely to have a large queue at the moment (though it may imply that the system did have a large queue until recently).

A Useful Approach

Given an ergodic Markov chain with transition probabilities p_{ij}. and a vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots)$ such that $\pi_i > 0$ and $\sum_{i=0}^{\infty} \pi_i = 1$ then if the scalars

$$p^*_{ij} = \frac{\pi_j p_{ji}}{\pi_i}, \tag{2}$$

form a stochastic matrix, that is

$$\sum_{j=0}^{\infty} p^*_{ij} = 1 \qquad \text{for all } i = 0, 1, \dots, \tag{3}$$

then $\boldsymbol{\pi}$ is the equilibrium distribution and the p^*_{ij} are the reversed transition probabilities. The proof of this is left as an exercise for the student. Note that this property holds even if the chain is not time reversible.

Therefore, if insight (or a lucky guess) provides such a $\boldsymbol{\pi}$ and p^*_{ij} then proving that equations (2) and (3) hold proves that the invariant density and reverse transition probabilities have been found.

Jackson's Theorem

Consider a network of K queues each with a single Poisson server. These queues are connected in a network so that on exiting one queue the customers may leave the network or join another queue. Customers enter the network at any queue as a Poisson process. On exiting any queue then they either move to a new queue (or possibly rejoin the same one) or leave the network entirely at random. Definitions:

P_{ij} the probability that a customer leaving i goes on to j.

r_j the arrival rate of new cusomers at queue j as a Poisson process.

λ_j is the total arrival rate at the queue j (all customers).

μ_j is the service rate of queue j.

$\rho_j = \lambda_j / \mu_j$ is the utilisation of queue j. This is assumed by hypothesis to be less than 1 for all queues.

It must be that:

$$\sum_{j=1}^{K} P_{ij} \le 1,$$

and this assumed to be a strict inequality (< 1) for at least one queue (there is at least one place where customers leave the system forever). It is also assumed that each customer entering the system can reach a queue which has a probability of leaving (and each customer will eventually leave the system almost surely). The probability that a customer leaving queue i leaves the system forever is given by:

$$\mathbb{P}\left[\text{Customer leaving queue } i \text{ leaves system}\right] = 1 - \sum_{j=1}^{K} P_{ij}.$$

The state of the system is represented by a vector $\mathbf{n} = (n_1, \ldots, n_K)$ where $n_i \in \mathbb{Z}_+$ is the number of customers in queue i. The system can now be viewed as a Markov chain with states in $\mathbb{Z}_+^K$. A special notation will be used to indicate transitions between states.

So, state $\mathbf{n}(j^+)$ is the state corresponding to a new arrival at queue j. That is

$$\mathbf{n}(j^+) = (n_1, \ldots, n_j + 1, \ldots, n_K).$$

The probability of such a transition is given by

$$p_{\mathbf{n}\mathbf{n}(j^+)} = r_j. \tag{4}$$

State $\mathbf{n}(j^-)$ similarly corresponds to customer leaving the system completely from state j. That is

$$\mathbf{n}(j^-) = (n_1, \ldots, n_j - 1, \ldots, n_K).$$

The probability of such a transition is given by

$$p_{\mathbf{n}\mathbf{n}(j^-)} = \mu_j (1 - \sum_i P_{ji}). \tag{5}$$

Finally, state $\mathbf{n}(i^+, j^-)$ corresponds to a customer leaving queue j and joining queue i. That is

$$\mathbf{n}(i^+, j^-) = (n_1, \ldots, n_i + 1, \ldots n_j - 1, \ldots, n_K).$$

The probability of such a transition is given by

$$p_{\mathbf{n}\mathbf{n}(i^+, j^-)} = \mu_j P_{ji}. \tag{6}$$

Let $\boldsymbol{\pi}(\mathbf{n}) = \boldsymbol{\pi}(n_1, \ldots, n_K)$ be the equilibrium probability of the state $\mathbf{n}$. By finding $\boldsymbol{\pi}(\mathbf{n})$ the probability of any given distribution is given and the expected queue lengths can be determined. Jackson's theorem is the remarkable claim that, under the conditions given, the queues can be treated as K independent M/M/1 queues.

Theorem 2. *Assuming that $\rho_j < 1$ for all j then for $n_1, \ldots n_k \in \mathbb{Z}_+$:*

$$\boldsymbol{\pi}(\mathbf{n}) = \pi_1(n_1)\pi_2(n_2) \ldots \pi_K(n_K), \tag{7}$$

where

$$\pi_j(n_j) = \rho_j^{n_j}(1 - \rho_j) \qquad \textit{for all } n_j \in \mathbb{Z}_+. \tag{8}$$

Proof. Assume without loss of generality that $\lambda_j > 0$ for all j[1].

Take two states, $\mathbf{n}$ and $\mathbf{n}'$. The probability of a transition between them is $p_{\mathbf{n}\mathbf{n}'}$. The reverse probabilities are denoted by $p^*_{\mathbf{n}\mathbf{n}'}$. (Note that the chain is not in general reversible.) Now, from

[1] If $\lambda_j = 0$ then $\pi_j(0) = 1$ and $\pi_j(n_j) = 0$ for $n_j > 0$. Hence this can be removed from equation (7).

equations (2) and 3, the theorem is proved, if given equations (7) and (8), then the probabilities given by

$$p^*_{\mathbf{n}\mathbf{n}'} = \frac{\boldsymbol{\pi}(\mathbf{n}')p_{\mathbf{n}'\mathbf{n}}}{\boldsymbol{\pi}(\mathbf{n})}, \tag{9}$$

satisfy

$$\sum_{\mathbf{n}'} p^*_{\mathbf{n}\mathbf{n}'} = 1, \tag{10}$$

for all $\mathbf{n}$

Note that from equation (9) then $p_{\mathbf{n}\mathbf{n}} = p^*_{\mathbf{n}\mathbf{n}}$ for all $\mathbf{n}$ and also that

$$\sum_{\mathbf{n}'} p_{\mathbf{n}\mathbf{n}'} = 1.$$

Therefore, equation (10) is equivalent to showing

$$\sum_{\mathbf{n}' \neq \mathbf{n}} p_{\mathbf{n}\mathbf{n}'} = \sum_{\mathbf{n}' \neq \mathbf{n}} p^*_{\mathbf{n}\mathbf{n}'}, \tag{11}$$

for all $\mathbf{n}$.

The following transitions have already been defined:

$$\begin{aligned} p_{\mathbf{n}\mathbf{n}(j^+)} &= r_j \\ p_{\mathbf{n}\mathbf{n}(j^-)} &= \mu_j(1 - \sum_i P_{ji}) \\ p_{\mathbf{n}\mathbf{n}(i^+j^-)} &= \mu_j P_{ji}. \end{aligned}$$

For all other $\mathbf{n} \neq \mathbf{n}'$ then

$$p_{\mathbf{n}\mathbf{n}'} = 0.$$

Therefore, for the forward system, for all $\mathbf{n}$ then

$$\begin{aligned} \sum_{\mathbf{n}' \neq \mathbf{n}} p_{\mathbf{n}\mathbf{n}'} &= \sum_{j=1}^{K} p_{\mathbf{n}\mathbf{n}(j^+)} + \sum_{j|n_j>0} p_{\mathbf{n}\mathbf{n}(j^-)} + \sum_{i,j|n_j>0} p_{\mathbf{n}\mathbf{n}(i^+,j^-)} \\ &= \sum_{j=1}^{K} r_j + \sum_{j|n_j>0} \mu_j(1 - \sum_{i=1}^{K} P_{ji}) + \sum_{i,j|n_j>0} \mu_j P_{ji}. \end{aligned}$$

Which finally gives

$$\sum_{\mathbf{n}' \neq \mathbf{n}} p_{\mathbf{n}\mathbf{n}'} = \sum_{j=1}^{K} r_j + \sum_{j|n_j>0} \mu_j. \tag{12}$$

Now, taking the reverse transitions, from equation (8) then

$$\boldsymbol{\pi}(\mathbf{n}(j^+)) = \rho_j \boldsymbol{\pi}(\mathbf{n}).$$

Similarly

$$\boldsymbol{\pi}(\mathbf{n}(j^-)) = \boldsymbol{\pi}(\mathbf{n})/\rho_j,$$

and also

$$\boldsymbol{\pi}(\mathbf{n}(i^+, j^-)) = \rho_i \boldsymbol{\pi}(\mathbf{n})/\rho_j.$$

From equation (9) and the above three equations then the reverse transitions are given by

$$p^*_{\mathbf{n}\mathbf{n}(j^+)} = \frac{\boldsymbol{\pi}(\mathbf{n}(j^+))}{\boldsymbol{\pi}(\mathbf{n})} p_{\mathbf{n}(j^+)\mathbf{n}} = \frac{\boldsymbol{\pi}(\mathbf{n}(j^+))}{\boldsymbol{\pi}(\mathbf{n})} p_{\mathbf{n}\mathbf{n}(j^-)} = \lambda_j(1 - \sum_i P_{ji})$$
$$p^*_{\mathbf{n}\mathbf{n}(j^-)} = \frac{\boldsymbol{\pi}(\mathbf{n}(j^-))}{\boldsymbol{\pi}(\mathbf{n})} p_{\mathbf{n}(j^-)\mathbf{n}} = \frac{\boldsymbol{\pi}(\mathbf{n}(j^-))}{\boldsymbol{\pi}(\mathbf{n})} p_{\mathbf{n}\mathbf{n}(j+1)} = \mu_j r_j / \lambda_j$$
$$p^*_{\mathbf{n}\mathbf{n}(i^+,j^-)} = p^*_{\mathbf{n}(j^+,i^-)\mathbf{n}} = \frac{\boldsymbol{\pi}(\mathbf{n})}{\boldsymbol{\pi}(\mathbf{n}(j^+,i^-))} p_{\mathbf{n}\mathbf{n}(j^+,i^-)} = \frac{\rho(i)}{\rho(j)} \mu_i P_{ij} = \mu_j \lambda_i P_{ji} / \lambda_j.$$

As before, for all other $\mathbf{n} \neq \mathbf{n}'$ then

$$p^*_{\mathbf{n}\mathbf{n}'} = 0.$$

Therefore, summing for all $\mathbf{n}$ for the reversed system

$$\begin{aligned}
\sum_{\mathbf{n}' \neq \mathbf{n}} p^*_{\mathbf{n}\mathbf{n}'} &= \sum_{j=1}^{K} p^*_{\mathbf{n}\mathbf{n}(j^+)} + \sum_{j|n_j>0} p^*_{\mathbf{n}\mathbf{n}(j^-)} + \sum_{j,i|n_j>0} p^*_{\mathbf{n}\mathbf{n}(i^+,j^-)} \\
&= \sum_{j=1}^{K} \lambda_j(1 - \sum_{i=1}^{K} P_{ji}) + \sum_{j|n_j>0} \frac{\mu_j r_j}{\lambda_j} + \sum_{j,i|n_j>0} \frac{\mu_j \lambda_i P_{ij}}{\lambda_j} \\
&= \sum_{j=1}^{K} \lambda_j(1 - \sum_{i=1}^{K} P_{ji}) + \sum_{j|n_j>0} \frac{\mu_j(r_j + \sum_{i=1}^{K} \lambda_i P_{ij})}{\lambda_j}.
\end{aligned}$$

This finally gives

$$\sum_{\mathbf{n}' \neq \mathbf{n}} p^*_{\mathbf{n}\mathbf{n}'} = \sum_{j=1}^{K} \lambda_j(1 - \sum_{i=1}^{K} P_{ji}) + \sum_{j|n_j>0} \mu_j. \tag{13}$$

Finally, summing all the processes entering queue j (either from another queue or from outside, and remembering that the exit rate of queue j must be equal to the input rate (if the queue is not to grow forever) then:

$$\lambda_j = r_j + \sum_{i=1}^{K} \lambda_i P_{ij},$$

for all $j = 1 \ldots K$. Rearranging this gives

$$r_j = \lambda_j - \sum_{i=1}^{K} \lambda_i P_{ij}.$$

Summing over all queues gives:

$$\sum_{j=1}^{K} r_j = \sum_{j=1}^{K} \lambda_j(1 - \sum_{i=1}^{K} P_{ji}). \tag{14}$$

Combining equations (12), (13) and (14) gives

$$\sum_{\mathbf{n}' \neq \mathbf{n}} p_{\mathbf{n}\mathbf{n}'} = \sum_{\mathbf{n}' \neq \mathbf{n}} p^*_{\mathbf{n}\mathbf{n}'},$$

which is equation (11) as required and hence the theorem is proved. □

Jackson's Theorem Example

Amnesia house is a house which deals with people who have trouble with remembering things. Forgetful people decide to leave the building as a Poisson process a rate λ. The queue to leave is a single server Poisson process with a rate μ_1. Unfortunately, on leaving, a proportion p of them remember something they have forgotten and must join a separate queue established for people who have forgotten something and wish to reenter. This is also a Poisson process with a rate μ_2. If the forgetful people are chosen at random (and people may forget multiple times) then find N_1 the average number of people queuing to leave and N_2 the average number of people queuing to get back in (assuming that μ_1 and μ_2 are sufficiently large that the system is ergodic). Find the total number trying to leave.

Define λ_1 as the input rate to the queue to leave and λ_2 as the input rate to the queue to reenter.

The rates to each queue are

$$\lambda_1 = \lambda + \lambda_2,$$

and

$$\lambda_2 = p\lambda_1.$$

Solving gives

$$\lambda_2 = \frac{\lambda p}{1-p},$$

and

$$\lambda_1 = \frac{\lambda}{1-p}$$

Therefore

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{\lambda}{(1-p)\mu_1},$$

and

$$\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{\lambda p}{(1-p)\mu_2}.$$

By Jackson's theorem the two queues are independent M/M/1 queues. Therefore

$$\mathbb{P}\left[n_1, n_2\right] = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2).$$

Also

$$N_1 = \frac{\rho_1}{1-\rho_1},$$

and

$$N_2 = \frac{\rho_2}{1-\rho_2}.$$

Hence

$$N = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2}.$$