Networks II – Worksheet Two

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Birth Death Processes and Queuing

Question 1. Consider the $M/M/m/m$ queue. That is an $M/M/m$ queue where, if all servers are busy then the customers are turned away. Model this as a birth-death process.

1. Write down the birth death coefficients λ_k and μ_k.
2. From the equations for the general birth death process show that the probability that a customer finds the system full is given by:

$$\pi_m = \frac{(\lambda/\mu)^m/m!}{\sum_{n=0}^{m}(\lambda/\mu)^n \frac{1}{n!}}$$

Answer 1.

1. $$\lambda_k = \begin{cases} \lambda & 0 \leq k < m \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_k = \begin{cases} k\mu & 1 \leq k \leq m \\ 0 & \text{otherwise} \end{cases}$$

2. From the general equations for a birth death process:

$$\pi_0 = \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1}$$

and therefore the answer follows.

Question 2. (From Bertsekas and Gallager Q 3.31) Consider the following (spurious) argument about the $M/G/1$ queue. When a customer arrives, the probability that another customer is being served is $\lambda\overline{X}$. Since the served customer has mean service time $\overline{X}$ then the average time to complete the service is $\overline{X}/2$. Therefore, the mean residual service time is $(\lambda\overline{X}^2)/2$. What is wrong with this argument?

Answer 2. In brief: $\overline{X^2} \neq \overline{X}^2$.

Question 3. Taken from Kleinrock problem 2.13.

Consider a system in which the birth rate decreases and the death rate increases as the number in the system k increases. That is:

$$\lambda_k = \begin{cases} (K-k)\lambda & k \leq K \\ 0 & k \geq K \end{cases}$$

$$\mu_k = \begin{cases} k\mu & k \leq K \\ 0 & k \geq K \end{cases}$$

Write down the differential-difference equations for $P_k(t) = Pr\{k \text{ in system at time } t\}$.

Answer 3. The equations you should get are given below.

For $k = 0$:

$$\frac{dP_0(t)}{dt} = \mu P_1(t) - K\lambda P_0(t)$$

For $0 < k < K$:

$$\frac{dP_k(t)}{dt} = \lambda(K-k+1)P_{k-1}(t) + (k+1)\mu P_{k+1}(t) - [(K-k)\lambda + k\mu]P_k(t)$$

For $k = K$:

$$\frac{dP_K(t)}{dt} = \lambda P_{K-1}(t) - K\mu P_K(t)$$

Question 4. Consider the general birth-death process as discussed in the lectures with the birth rate λ_i for $i \geq 0$ and the death rate μ_i for $i \geq 1$. Assuming that the system is ergodic, prove the relation

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i},$$

where π_i is the equilibrium probability of the ith state.

Hint: Proof by induction is a good approach.

Answer 4. First set up the balance equations. State zero gives,

$$\pi_0 = (1-\lambda_0)\pi_0 + \mu_1\pi_1.$$

Rearranging gives

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0,$$

which is the equation we are trying to prove for $k = 1$.

The balance equation for some state i gives

$$\pi_i = \lambda_{i-1}\pi_{i-1} + (1-\lambda_i-\mu_i)\pi_i + \mu_{i+1}\pi_{i+1},$$

and a similar rearrangement for $i = 1$ gives

$$\pi_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}.$$

The equation rearranges to

$$\lambda_i \pi_i + \mu_i \pi_i = \lambda_{i-1} \pi_{i-1} + \mu_{i+1} \pi_{i+1}.$$

Assume that the induction hypothesis is true for i and $i - 1$ then,

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} = \frac{\lambda_{i-1}}{\mu_i} \pi_{i-1}.$$

Substituting this in the previous equation gives,

$$\lambda_i \pi_i + \lambda_{i-1} \pi_{i-1} = \lambda_{i-1} \pi_{i-1} + \mu_{i+1} \pi_{i+1},$$

Rearranging gives

$$\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}} \pi_i,$$

which is the relation desired.

Question 5. Consider the $M/M/1/2$ process with birth rate λ and death rate μ where $\mu > \lambda$. The final figure 2 means that at most two customers are allowed in the system — further customers arriving are turned away. This can be modelled as a birth-death process with the following characteristics:

$$\lambda_i = \begin{cases} \lambda & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_i = \begin{cases} \mu & i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

1. Represent the process as a Markov Chain — give the transition matrix $\mathbf{P}$.
2. If $P_i(t)$ is the probability that the process is in state i at time t then derive the three differential difference equations $\frac{dP_i(t)}{dt}$ for the system.
3. Write down a matrix equation which relates the three differential difference equations. Show how this relates to the transition matrix. (Hint: Your matrix equation should have the form):

$$\begin{bmatrix} \frac{dP_0(t)}{dt} \\ \frac{dP_1(t)}{dt} \\ \frac{dP_2(t)}{dt} \end{bmatrix} = \mathbf{X} \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{bmatrix}$$

4. Using the general solution for the Birth-Death Process from lectures, find $\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$.

Answer 5. 1.

$$\mathbf{P} = \begin{bmatrix} 1-\lambda & \lambda & 0 \\ \mu & 1-\lambda-\mu & \lambda \\ 0 & \mu & 1-\mu \end{bmatrix}$$

2. You should get:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

$$\frac{dP_1(t)}{dt} = (-\lambda - \mu)P_1(t) + \lambda P_0(t) + \mu P_2(t)$$

$$\frac{dP_2(t)}{dt} = -\mu P_2(t) + \lambda P_1(t)$$

3. They are related by:

$$\begin{bmatrix} \frac{dP_0(t)}{dt} \\ \frac{dP_1(t)}{dt} \\ \frac{dP_2(t)}{dt} \end{bmatrix} = (\mathbf{P} - \mathbf{I})^{\mathbf{T}} \begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{bmatrix}$$

4.

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}$$

Gives:

$$\pi_0 = \frac{1}{[1 + \rho + \rho^2]}$$

And combining with :

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$$

we get:

$$\pi_1 = \frac{\rho}{[1 + \rho + \rho^2]}$$

and

$$\pi_2 = \frac{\rho^2}{[1 + \rho + \rho^2]}$$

Question 6. Consider the $M/G/1$ queue where customers are waiting to pick up packages in the post office. Customers arrive in a Poisson process with an average rate of one every two minutes. Each customer has to pick up k packages ($0 < k < 4$). With $Pr\{k = 1\} = 0.5$, $Pr\{k = 2\} = 0.25$, $Pr\{k = 3\} = 0.2$ and $Pr\{k = 4\} = 0.05$. If the post office takes one minute to find each package, then use the P-K formula to find W the average waiting time in the queue.

Answer 6. $E[k] = (0.5) + 2(0.25) + 3(0.2) + 4(0.05) = 1.8$

$\overline{X} = 1.8$ minutes

$E[k^2] = [0.5 + 4(0.25) + 9(0.2) + 16(0.05)] = 4.1$

$\overline{X^2} = 4.1$ minutes^2

$\lambda = 1/2$ minutes^{-1}

The P-K formula can be written as:

$$W = \frac{\lambda \overline{X^2}}{2(1 - \lambda \overline{X})}$$

$W = \frac{0.5(4.1)}{2[1-0.5(1.8)]} = 10.25$ minutes.

Question 7. Consider the network given by Figure 1.

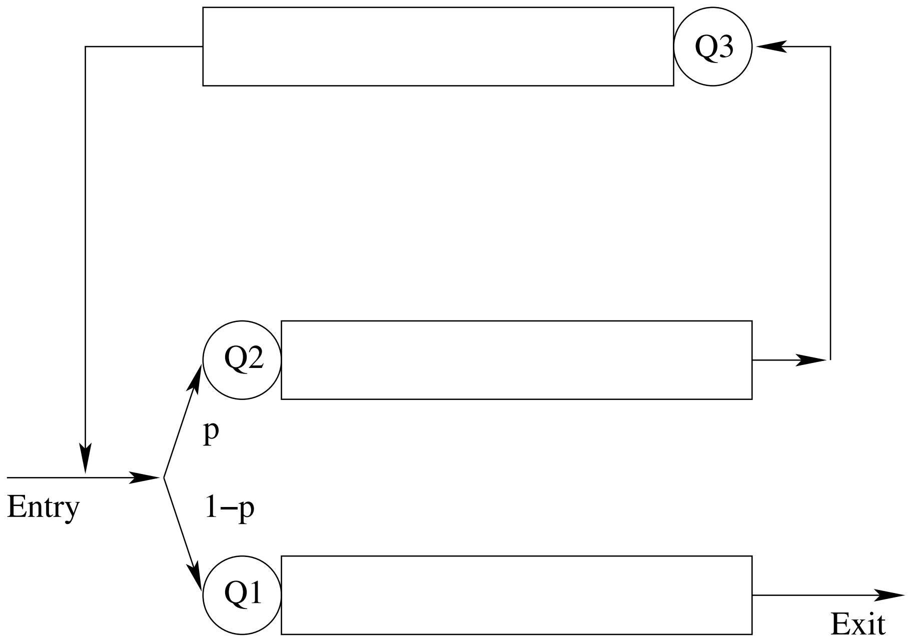

Figure 1: A simple network of queues

Packet traffic enters the network as a Poisson process with rate λ. The three queues all have a service rate μ where $\mu > \lambda$. When the traffic enters the system then, with probability p it enters Q2 and with probability $1 - p$ it enters Q1. Traffic leaving Q1 leaves the system. Traffic leaving Q2 enters Q3. Traffic leaving Q3 enters either Q1 or Q2 with probabilities $1 - p$ and p respectively. (The probability for each packet is independent).

1. Assuming that the system is such that all queues are ergodic calculate N_1, N_2 and N_3 the expected queue lengths at each queue. Calculate also T_1, T_2 and T_3 the expected time that a packet will spend in each queue before leaving it.
2. Therefore calculate N the expected total number of packets in the whole system and T the expected total time a packet spends in the system. Why is T not $T1 + T2 + T3$?
3. Find what value of p will overload the system in terms of λ and μ.

Answer 7. Let λ_1, λ_2 and λ_3 be the entry rates into Q1, Q2 and Q3 respectively and let ρ_1, ρ_2 and ρ_3 be the respective utilisations.

1. Clearly $\lambda_1 = (1 - p)\lambda + (1 - p)\lambda_3$ and $\lambda_2 = \lambda_3 = p\lambda + p\lambda_3$. Rearranging gives $\lambda_2 = \lambda_3 = p\lambda/(1 - p)$ and therefore $\lambda_1 = \lambda$.

Therefore

$$N_1 = \frac{\rho_1}{1-\rho_1} = \frac{(1-p)\lambda/\mu}{1-(1-p)\lambda/\mu} = \frac{(1-p)\lambda}{\mu-(1-p)\lambda},$$

and from Little's Theorem,

$$T_1 = \frac{1}{\mu-(1-p)\lambda}.$$

Similarly

$$N_1 = N_2 = \frac{\rho_2}{1-\rho_2} = \frac{p\lambda/(1-p)\mu}{1-p\lambda/(1-p)\mu} = \frac{p\lambda}{(1-p)\mu-p\lambda}.$$

and hence,

$$T_2 = T_3 = \frac{(1-p)}{(1-p)\mu-p\lambda}.$$

2. For the whole system $N = N_1 + N_2 + N_3$ and hence

$$N = \frac{(1-p)\lambda}{\mu-(1-p)\lambda} + \frac{2p\lambda}{(1-p)\mu-p\lambda},$$

therefore from Little's Theorem

$$T = \frac{(1-p)}{\mu-(1-p)\lambda} + \frac{2p}{(1-p)\mu-p\lambda}.$$

But

$$T_1 + T_2 + T_3 = \frac{1}{\mu-(1-p)\lambda} + \frac{2(1-p)}{(1-p)\mu-p\lambda}.$$

The two are different because T is the time an average packet spends in the system. It may go around the system multiple times or it may never go through $T2$ and $T3$ at all.

3. The system will overload when any of the quantities N_1, N_2 or N_3 become infinite. Since $\mu > \lambda$ this can never happen for N_1 but will happen for N_2 and N_3 when the denominator goes to zero. This happens as p increases such that $(1-p)\mu - p\lambda = 0$. This implies $\mu - p\mu - p\lambda = 0$ or $p = \mu/(\lambda+\mu)$.

Basic Graph Theory and Routing

Question 8. Show the steps of the Prim-Dijkstra algorithm beginning at O to create an MST for the graph.

Answer 8. Connect:

- $O \to C$
- $C \to E$
- $E \to A$
- $E \to F$
- $A \to B$

Question 9. Use Kruskal's Algorithm to create an MST for the same graph. In what order are nodes connected? (Indicate any possible ambiguities.) Why can Kruskal's algorithm not be used in a distributed manner here? Is the MST unique?

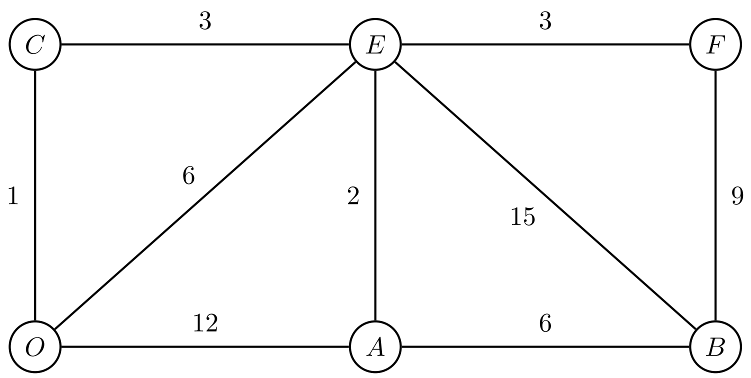

Figure 2: Figure for testing Prim-Dijkstra vs Kruskal's Algorithm. (Taken from B & G)

Answer 9. Connect

- $O \to C$
- $E \to A$
- $E \to C$ or
- $E \to F$ (order is ambiguous here)
- $A \to B$

Two arcs have a weight of three and therefore the distributed version of Kruskal's algorithm is not appropriate since there may not be a unique MST. However, the MST constructed is unique as can be seen if the steps were reversed in the above construction the MST is the same.

Question 10. The versions of Dijkstra's and Bellman-Ford's Algorithm in the lectures proved the algorithms for finding the shortest path from one origin to every destination on the network. Test your understanding of the proofs by constructing the reverse algorithms and proving them. That is, find the shortest paths from any origin to one destination.

Answer 10. The difference is, in fact, trivial and the reverse algorithms can be found in Bertsekas and Gallager pages 396 (Bellman-Ford) and 401 (Dijkstra).

Question 11. Use Bellman-Ford to find the shortest path from 1 to each other node in the diagram. Show all values of D_i^j in your working. After how many steps is the algorithm complete?

Answer 11. The values are as shown in this table:

i	D_1^i	D_2^i	D_3^i	D_4^i	D_5^i	D_6^i
1	0	3	5	∞	∞	∞
2	0	3	4	7	13	∞
3	0	3	4	7	9	12
4	0	3	4	7	9	10
5+	0	3	4	7	9	10

Question 12. Use Dijkstra's algorithm to find the shortest paths from node 1 in the same diagram. List the permanent nodes and the temporary nodes (and costs) at each step of the algorithm. Therefore, what is the shortest path to node 6.

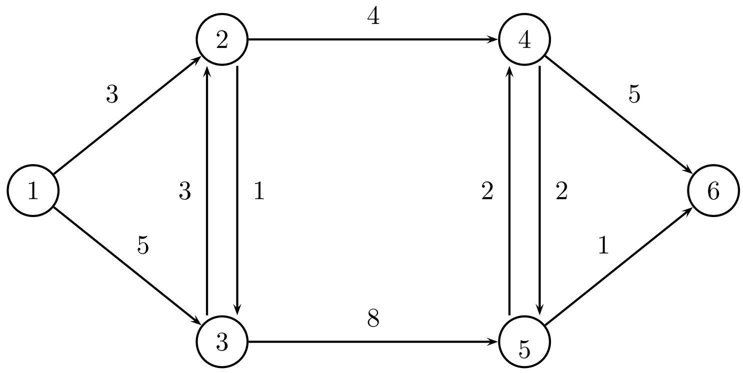

Figure 3: Weighted graph for Dijkstra's algorithm and Bellman-Ford

Permanent nodes	Temporary Nodes
1 (0)	2 (3), 3 (5)
1(0), 2(3)	3(4), 4(7)
1(0), 2(3), 3(4)	4(7), 5(12)
1(0), 2(3), 3(4), 4(7)	5(9), 6(12)
1(0), 2(3), 3(4), 4(7), 5(9)	6(10)
1(0), 2(3), 3(4), 4(7), 5(9), 6(10)	

Table 1: Nodes for Dijkstra's Algorithm

Answer 12. Nodes are shown with the costs after them in brackets.

$1 \to 2 \to 4 \to 5 \to 6$ total cost 10.