Matrix Example

Question

Given the Markov chain showing the transition between states two states (0 and 1):

$$\mathbf{P} = \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \tag{1}$$

Find (a) The equilibrium probabilities and (b) The probability that if it starts in state 0 it is in state 0 after n transitions.

Answer

At equilibrium we have $\pi_i = \sum_j p_{ji}\pi_j$. In this case we have:

$$\pi_0 = 2/3.\pi_0 + 1/2.\pi_1 \tag{2}$$

Since we know that $\pi_0 + \pi_1 = 1$ we get the solution:

$\pi_0 = 3/5$ and $\pi_1 = 2/5$.

In general, it is worthwhile checking these values by substituting them into the other balance equation:

$$\pi_1 = 1/3.\pi_0 + 1/2.\pi_1 \tag{3}$$

which indeed yields an equality with the above values.

In order to calculate the matrix $\mathbf{P}^n$ we need to find the eigenvalues. Therefore we need to solve:

$$\begin{vmatrix} 2/3-\lambda & 1/3 \\ 1/2 & 1/2-\lambda \end{vmatrix} = 0 \tag{4}$$

This gives:

$$(2/3-\lambda)(1/2-\lambda) - 1/6 = 0 \tag{5}$$

It is left as an exercise to the student to verify that this gives two eigenvalues: $\lambda_0 = 1$ and $\lambda_1 = 1/6$.

We therefore have:

$$\mathbf{P} = \mathbf{U}^-1 \begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix} \mathbf{U} \tag{6}$$

for some invertible matrix $\mathbf{U}$. We do not need to find this matrix but this tells us that all expressions $p_{ij}^{(n)}$ are of the form:

$$p_{ij}^{(n)} = a1^n + b(1/6)^n = a + b(1/6)^n \tag{7}$$

for some constants a and b which usually have different values for each i, j pair.

Trivially $p_{00}^{(0)} = 1$ therefore for the $0 \to 0$ transition:

$$1 = a + b \tag{8}$$

Again, trivially $p_{00}^{(1)} = 2/3$ and therefore:

$$2/3 = a + b/6 \tag{9}$$

combining these gives us values $a = 3/5$ and $b = 2/5$ (left as a simple exercise for student). Therefore:

$$p_{00}^{(n)} = 3/5 + 2/5.(1/6)^n \tag{10}$$

A nice check is availble on our work here by calculating $p_{00}^{(2)}$ directly using the matrix. Two possibilities — that the chain remains in state 0 or that it goes to state 1 and returns immediately. Therefore, by inspection we can say:

$$p_{00}^{(2)} = (2/3)^2 + 1/3.1/2 = 4/9 + 1/6 = 11/18 \tag{11}$$

we wish to check this is equal to:

$$p_{00}^{(2)} = 3/5 + 2/5(1/6)^2 = 3/5 + 2/5.1/36 \tag{12}$$

which is 11/18 as expected. This gives us reasonable confidence that we made no numerical errors in our calculations.