0580020

UNIVERSITY OF YORK

MSc Examinations 2004
MATHEMATICS
Networks

Time Allowed: 3 hours.

Answer 4 questions.
Standard calculators will be provided but should be unnecessary.

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1. Consider a network with n directed arcs and N routes joining α to β. Let A be the arc-route incidence matrix (so that $A_{ar} = 1$ if arc a lies on route r and $A_{ar} = 0$ otherwise), let x_a be the flow along arc a (for each arc a) and let x be the n-vector of these arc flows. Also let the cost of traversing arc a be $c_a(x)$; where

$$c_a : \mathbb{R}_+^n \to \mathbb{R}_+$$

is (for each a) a given smooth function; and, for each x, let $c(x)$ denote the n-vector whose a^{th} coordinate is $c_a(x)$.

(i) Let $X \in \mathbb{R}_+^N$ denote an arbitrary route-flow vector. Show how to calculate $x_a = x_a(X)$, the flow along arc a which arises from the route-flow vector X.

Hence write down a formula, involving the incidence matrix A, for the arc-flow vector x which arises from a route-flow vector X.

(5 marks)

(ii) For an arbitrary route-flow vector X, show how to calculate the cost $C_r(X)$ of travel along route r in terms of the cost flow functions $c_a(\cdot)$.

Hence write down a function $C : \mathbb{R}_+^N \to \mathbb{R}_+^N$ determining the N-vector $C(X)$ of route costs in terms of the route-flow vector X, the arc cost-flow function $c(\cdot)$ and the incidence matrix A.

(5 marks)

(iii) Suppose that the arc cost-flow function $c : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is monotone. Show that the corresponding route-cost function $C : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is monotone.

(7 marks)

(iv) Consider the network shown on the opposite page; where all the arc cost functions are identities. That is (for each arc) if the arc flow along arc a is x_a then the cost of travel along arc a is also x_a.

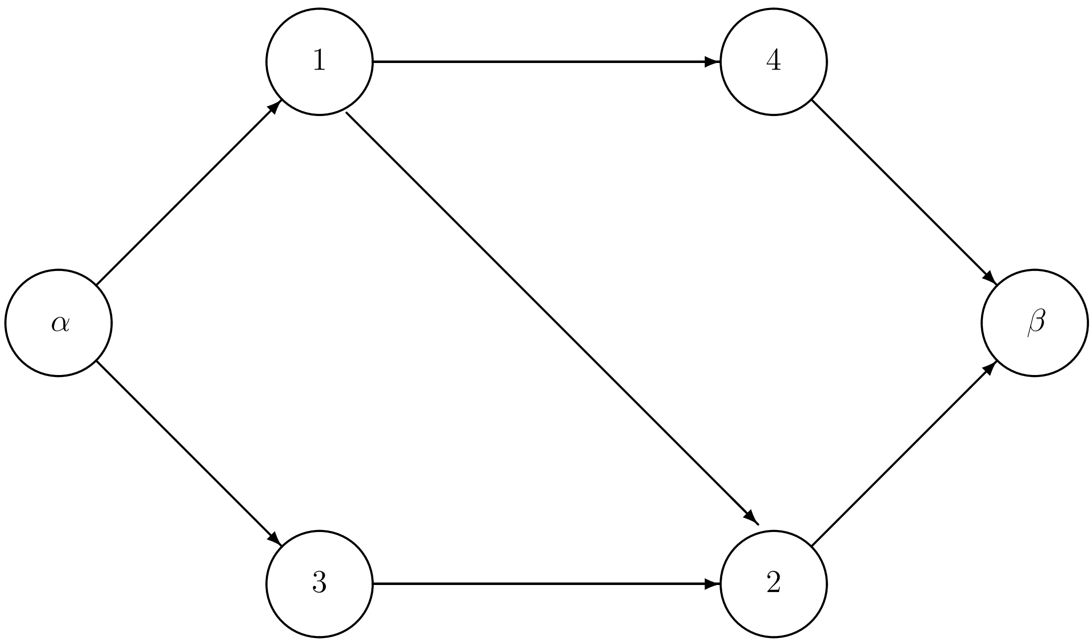
For each of the three routes joining α to β determine the cost of traversing that route in terms of the flows X_1, X_2, X_3 along the three routes. Hence, for this network, write down the cost vector $C(X)$ in terms of the route-flow vector X.

(4 marks)

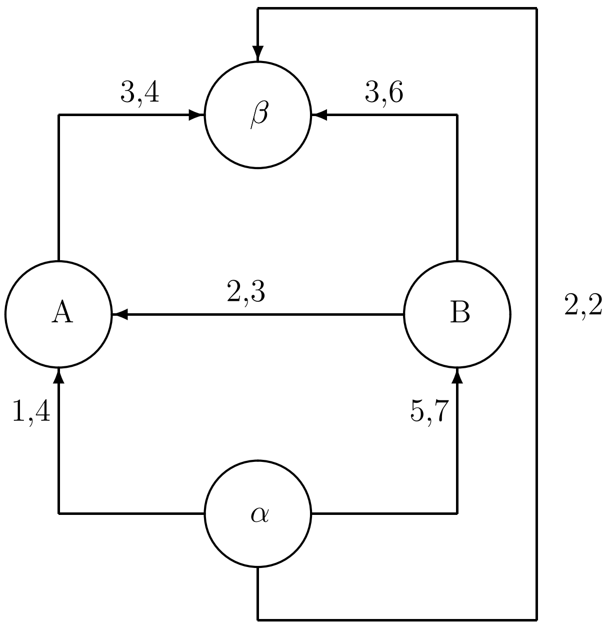
(v) By direct calculation (and without quoting the result proved in (iii) above), verify that the route cost-flow function $C(\cdot)$ determined in answer to part (iv) above is monotone.

(4 marks)

Network for question 1



Network for question 2, part A



2. PART A

(i) For the network shown above, an initial flow along each arc and the capacity of that arc are marked. Perform one iteration of the labelling algorithm to increase the flow shown from α to β.

(6 marks)

(ii) Show a sequence of flow patterns on the network terminating at a maximal flow which arises from repeated application of the labelling algorithm. (In this section you are not asked to perform each labelling, just the results of applying the labelling procedure until a maximal flow is attained.)

(6 marks)

(iii) Explain why the flow reached in part (ii) above is maximal.

(3 marks)

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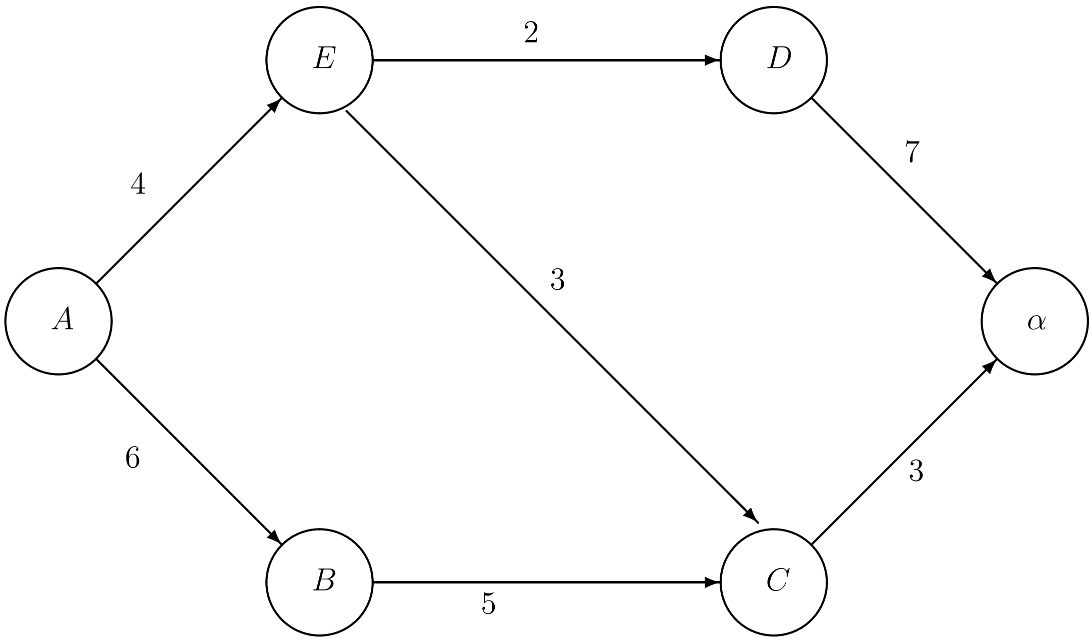
Network for question 2 part B



PART B

To calculate the lengths of the shortest paths from all nodes to a single node in a network Dijkstra's algorithm maintains and updates (a) a set P of permanently labelled nodes, (b) a set T of temporarily labelled nodes, (c) the permanent labels and (d) the temporary labels.

Consider the directed network above; where arc lengths are marked. For this network, explain how Dijkstra's algorithm may be used to calculate the least distances from the nodes A, B, C, D, E to the node α; indicating the sets P and T at each iteration and the labels attached to the nodes in these two sets at each iteration.

(10 marks)

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3. Consider a network with just two non-intersecting routes (route 1 and route 2) joining an origin α to a destination β. Suppose that route 1 has a continuous non-decreasing cost-flow function $F_1 : \mathbb{R}_+ \to \mathbb{R}_+$, that route 2 has a continuous non-decreasing cost-flow function $F_2 : \mathbb{R}_+ \to \mathbb{R}_+$, that $F_1(1) > F_2(0)$, and that $F_1(\frac{1}{2}) = F_2(\frac{1}{2})$.

Suppose that a route-flow vector $X(t)$ depends continuously on time t (for $t \geq 0$) and also satisfies the following conditions.

The initial (time zero) flow vector $X(0) = (X_1(0), X_2(0))$ satisfies:

$$X_1(0) = 1 \text{ and } X_2(0) = 0.$$

Also, for all $t > 0$,

$$dX_1(t)/dt = -[F_1(X_1(t)) - F_2(X_2(t))]$$

and

$$dX_2(t)/dt = -[F_2(X_1(t)) - F_1(X_2(t))].$$

(i) For all (X_1, X_2) such that $F_1(X_1) > F_2(X_2)$, let

$$V(X) = X_1\{[F_1(X_1) - F_2(X_2)]\}.$$

Show that $d[V(X(t))]/dt < 0$ for all $t > 0$.

(5 marks)

[You may assume here that $F_1(X_1(t)) > F_2(X_2(t))$ for all $t \geq 0$.]

(ii) Illustrate this result by drawing and commenting on a figure which demonstrates that $V(X(t))$ declines as time passes.

(5 marks)

(iii) Show how to define an objective function W such that

$$grad\ W(X) = F(X)$$

for all $X = (X_1, X_2)$ such that $X_1 \geq 0$ and $X_2 \geq 0$. Using your definition of W, prove that $d[W(X(t))]/dt < 0$ for all $t > 0$.

(5 marks)

[You may assume here that $F_1(X_1(t)) > F_2(X_2(t))$ for all $t \geq 0$.]

(iv) Illustrate this result by drawing and commenting on a figure which demonstrates that $W(X(t))$ declines as time passes.

(5 marks)

(v) Does it follow from the conditions on F and X that $F_1(X_1(t)) > F_2(X_2(t))$ for all $t \geq 0$? Justify your answer.

(5 marks)

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4. Consider the $M/G/1$ queuing system. Assume that customer service times are independent and identically distributed.

Let X_i be the service time of the ith customer and X be the time series of these service times.

The mean service time is $\overline{X} = E[X] = 1/\mu$ (where μ is the mean service rate).

The second moment of service time is $\overline{X^2} = E[X^2]$.

Let R_i be the residual service time (the service time remaining for the customer currently being served) when the ith customer arrives at the queue. (If the queue is empty when the ith customer arrives then $R_i = 0$.)

Let $r(\tau)$ be the residual service time (the service time remaining for the customer currently being served) at time τ.

If t is a time where the system is empty then define $M(t)$ as the number of customers served by time t.

(i) Draw a graph of $r(\tau)$ versus τ. Mark on your graph X_i, t and $M(t)$.
(3 marks)

(ii) Assume that:

- The mean residual time $R = lim_{i\to\infty} E[R_i]$ where $R = \lim_{t\to\infty} \frac{1}{t}\int_0^t r(\tau)d\tau$.
- $\lim_{t\to\infty} \frac{M(t)}{t} = \lambda$ where λ is the mean arrival rate.

From your diagram from the previous part show that

$$R = \frac{1}{2}\lambda\overline{X^2}.$$

(10 marks)

(iii) Let N_Q be the mean number of customers in the queue. The mean waiting time in the queue, W is given by

$$W = R + \frac{1}{\mu}N_Q.$$

From this and your previous answer, prove the Pollaczek-Khinchin formula

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)},$$

where $\rho = \lambda/\mu$.
(2 marks)

(iv) A system with a slow output line queues data packets. The data packets arrive with a Poisson distribution at a rate λ and have lengths which are independent and identically distributed. The lengths are flatly distributed between s (shortest) and l (longest) bytes with $s, l \in \mathbb{N} : s \leq l$. When it is free, the system sends the first packet in the queue and takes n ms ($n/1000$ seconds) to send a packet of length n bytes.

Calculate λ_m the maximum sending rate in packets/second which gives the system 100% utilisation $\rho = 1$.

(3 marks)

(v) Let the actual arrival rate $\lambda = \lambda_m/2$. Show that the average waiting time W for a packet is given by

$$W = \frac{l(l+1)(2l+1) - (s-1)s(2s-1)}{1000(l+s)(1+l-s)} \text{ seconds.}$$

You may find the equation

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

useful.

(7 marks).

5. Consider a general birth-death process with birth rates λ_i and death rates μ_i.

(i) The M/M/m/m server system has m Poisson servers each working at a rate μ and the system has an arrival rate of λ. When all servers are full customers are turned away. Write down the birth and death rates of such a system.
(2 marks)

(ii) Draw the Markov chain for a general birth death process. Give the transition matrix $\mathbf{P}$ showing at least p_{ij} for all $i, j \leq 3$.
(5 marks)

(iii) Let π_i be the equilibrium probability of state i and assume that the chain is ergodic. Give the balance equations for states zero and one of the chain. Rearrange these to get expressions for π_1 and π_2 in terms of π_0 only.
(5 marks)

(iv) Show by induction (or otherwise) that

$$\pi_n = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} \pi_0.$$

(8 marks)

(v) For the $M/M/m/m$ system find, in terms of π_0 and ρ (the utilisation), the probability that, when a randomly chosen customer arrives the customer is turned away.
(5 marks)

6. (i) Define the terms *tree*, *subtree*, *minimum weight spanning tree* (MST) and *fragment*.

(4 marks)

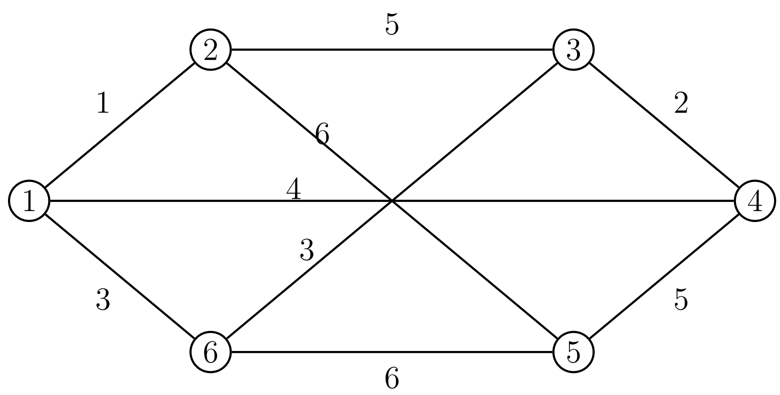
(ii) Prove the following proposition:

Let F be a fragment of a graph G. Let $\alpha = (i, j)$ be a the minimum weight arc in G such that $i \in F$ and $j \notin F$. If F' is constructed by adding the node j and the arc α to F then F' is also a fragment.

You may use without proof the fact that the number of arcs in a tree is given by $A = N - 1$ where A is the number of arcs and N is the number of nodes.

(6 marks)

(iii) Define the Prim-Dijkstra algorithm. Assuming it begins at node 1 in the figure below, list the order in which it adds arcs and draw the final MST.



(5 marks)

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(iv) Consider the Markov chain with states numbered 1 and 2 and the following transition matrix

$$\mathbf{P} = \begin{bmatrix} 2/5 & 3/5 \\ 1/3 & 2/3 \end{bmatrix}$$

- Calculate the eigenvalues of $\mathbf{P}$ and hence give the general form for $p_{ij}^{(n)}$ (the n step transition probability from state i to state j).
- From inspection of the matrix calculate the values of $p_{11}^{(0)}$ and $p_{11}^{(1)}$ and hence (or otherwise) calculate $p_{11}^{(n)}$. Check your answer by calculating $p_{11}^{(2)}$ directly from the matrix (showing all working).
- From your expression for $p_{11}^{(n)}$ (or by any other means) calculate the equilibrium probabilities π_1 and π_2.

(10 marks)

1.

2.

3.

4. (i) Three marks for some easy book work. Figure 1 shows residual times for the $M/G/1$ queuing system.

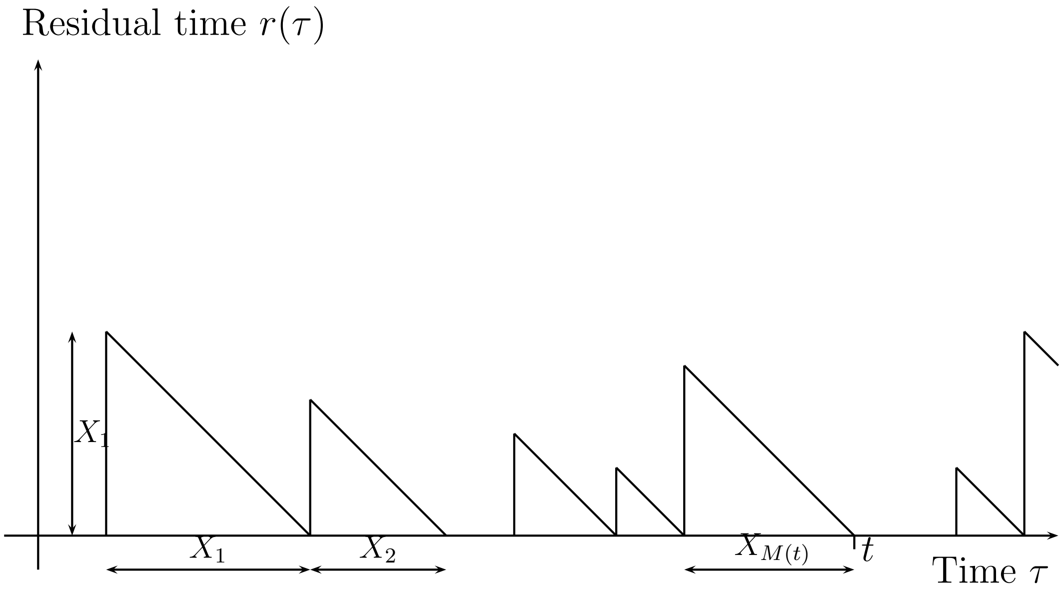


Figure 1: Service Time of Arrivals at an $M/G/1$ queue.

(ii) Again, simple book work. Expressing the area under the graph in two different ways

$$\frac{1}{t}\int_0^t r(\tau)d\tau = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2}X_i^2$$

Taking limits as $t \to \infty$ then

$$R = \lim_{t\to\infty} \frac{M(t)}{t} \frac{\sum_{i=1}^{M(t)} X_i^2}{2M(t)}.$$

Now, since the output rate must be the input rate λ then

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lambda.$$

Substituting above gives the required formula

$$R = \frac{1}{2}\lambda\overline{X^2}.$$

(iii) From Little's Theorem $N_Q = W\lambda$. Therefore

$$W = \frac{1}{2}\lambda\overline{X^2} + W\rho.$$

leading to the required answer

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)}.$$

(iv) The average packet transmission time is

$$\overline{X} = \frac{l+s}{2} \text{ ms } = \frac{l+s}{2000} \text{ sec },$$

and therefore since $\mu = 1/\overline{X}$ and $\rho = \lambda/\mu$ then

$$1 = \lambda_m \frac{l+s}{2000} \text{seconds.}$$

Hence

$$\lambda_m = \frac{2000}{l+s} \text{ packets/second }.$$

(v) The second moment of service time is given by

$$\overline{X^2} = \frac{\sum_{i=s}^{l} i^2}{1000^2(1+l-s)} = \frac{\sum_{i=0}^{l} i^2 - \sum_{i=0}^{s-1} i^2}{1,000,000(1+l-s)}.$$

From the given equation

$$\overline{X^2} = \frac{l(l+1)(2l+1) - (s-1)s(2s-1)}{1,000,000(1+l-s)}.$$

Therefore, from the P-K equation with $\lambda = 1000/(l+s)$ and $\rho = 1/2$ then

$$W = \frac{l(l+1)(2l+1) - (s-1)s(2s-1)}{1000(l+s)(1+l-s)} \text{ seconds.}$$

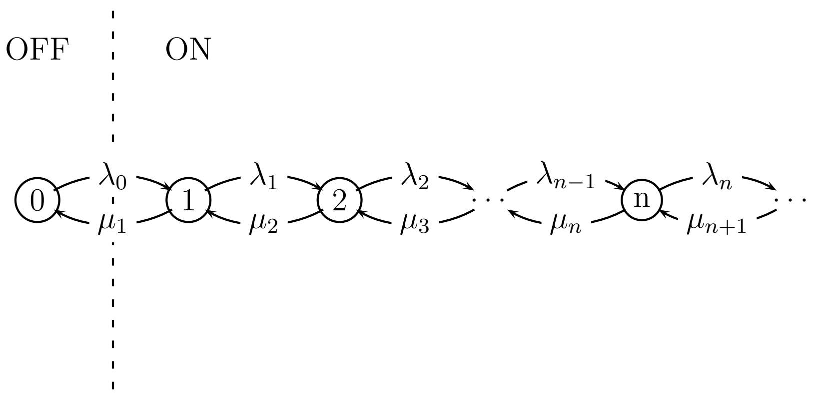
5. (i) A simple question for weak students to pick up some marks.
The death rates are given by

$$\mu_i = \begin{cases} i\mu & 0 \le i \le m \\ 0 & \text{otherwise} . \end{cases}$$

The birth rates are given by

$$\lambda_i = \begin{cases} \lambda & 0 \le i < m \\ 0 & \text{otherwise} . \end{cases}$$

(ii) The diagram is shown below.



The transition matrix is given by

$$\mathbf{P} = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & 0 & 0 & \dots & \\ \mu_1 & 1-\lambda_1-\mu_1 & \lambda_1 & 0 & \dots & \\ 0 & \mu_2 & 1-\lambda_2-\mu_2 & \lambda_2 & & \dots \\ 0 & 0 & \mu_3 & 1-\lambda_3-\mu_3 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

(iii) The balance equation for state zero is

$$\pi_0 = (1-\lambda_0)\pi_0 + \mu_1\lambda_1,$$

which rearranges to

$$\pi_1 = \pi_0 \frac{\lambda_0}{\mu_1}.$$

The balance equation for state one is

$$\pi_1 = (1-\lambda_1-\mu_1)\pi_1 + \lambda_0\pi_0 + \mu_2\pi_2,$$

Substituting for π_1 and rearranging gives

$$\begin{aligned}\mu_2\pi_2 &= (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1}\pi_0 - \lambda_0\pi_0 \\ \pi_2 &= \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0.\end{aligned}$$

(iv) This is new work (which was suggested as an exercise for students in class) but should not be too challenging. The case for π_0 has already been shown. Assume that

$$\pi_n = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0.$$

The balance equation for state n is

$$\pi_n = (1 - \lambda_n - \mu_n)\pi_n + \lambda_{n-1}\pi_{n-1} + \mu_{n+1}\pi_{n+1}.$$

Rearranging and substituting our assumption gives

$$\begin{aligned}\pi_{n+1}\mu_{n+1} &= (\lambda_n + \mu_n)\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0 - \lambda_{n-1}\prod_{i=1}^{n-1} \frac{\lambda_{i-1}}{\mu_i}\pi_0 \\ &= (\lambda_n + \mu_n)\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0 - \mu_n\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0 \\ &= \lambda_n\prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0 \\ \pi_{n+1} &= \prod_{i=1}^{n+1} \frac{\lambda_{i-1}}{\mu_i}\pi_0,\end{aligned}$$

which completes the proof by induction.

(v) This is simply asking for π_m. The utilisation is $\rho = \lambda/m\mu$. From the above equation

$$\pi_m = \prod_{i=1}^{m} \frac{\lambda}{i\mu}\pi_0 = \prod_{i=1}^{m} \frac{m\rho}{i}\pi_0 = \frac{(m\rho)^m}{m!}\pi_0.$$

6. (i) Tree: A connected graph with no cycles.

A subtree is a sub graph of G which is a tree. (A subgraph is a subset of nodes and arcs from a graph which is itself a graph).

A Minimum Weight Spanning Tree of a weighted graph G is a tree containing all the nodes of G such that no tree containing all nodes of G can be found where the sum of the arc weights is smaller.

A fragment is a subtree of an MST.

(ii) Let M be the MST of which F is a subtree. Choose some $\alpha = (i, j)$ as described in the question. If $\alpha = (i, j) \notin M$ (where α is the minimum weight arc as described above) then there must be a cycle formed by the arcs of M and α (since it has one more arc than is necessary to be a tree). Since $j \notin F$ by definition, then there must exist some arc $\beta = (k, l) \in M$ where $\beta \neq \alpha$ which is part of the cycle and where $k \in F$ and $l \notin F$. Deleting β from M and adding α instead must therefore result in a spanning tree M' (since M' is a subgraph of G which has no cycles and has the same number of nodes and arcs as M). Since α has lower weight than β then M' has a lower total weight than M which is a contradiction. Therefore, our original assumption that $\alpha = (i, j) \notin M$ cannot be true and the proposition is proved.

(iii) The Prim-Dijkstra algorithm takes a starting fragment of a single node and adds arcs as described in the proposition until an MST is formed. Starting at node 1, in the given graph it would add $(1, 2)$, $(1, 6)$, $(6, 3)$, $(3, 4)$ and $(4, 5)$ which is an MST.

(iv) The eigen values are the solutions to:

$$(2/5 - \lambda)(2/3 - \lambda) - (3/5)(1/3) = 0.$$

This gives $\lambda_1 = 1$ and $\lambda_2 = 1/15$.
The general form for the n step probability is therefore

$$p_{ij}^{(n)} = A + B(1/15)^n.$$

From the matrix

$$p_{11}^{(0)} = 1,$$

and

$$p_{11}^{(1)} = 2/5.$$

Hence $B = 9/14$ and $A = 5/14$ and, in general

$$p_{11}^{(2)} = 5/14 + 9/14(1/15)^n.$$

From the matrix itself

$$p_{11}^{(2)} = 2/5.2/5 + 3/51/3 = 4/25 + 1/5 = 9/25.$$

From the general expression

$$\begin{aligned} p_{11}^{(2)} &= 5/14 + 9/14.1/(15^2) \\ &= (5.15.15 + 9)/(15.15.14) = 1134/15.15.14 = 9/25 \end{aligned}$$

As $n \to \infty$ then $p_{11}^n \to 5/14$ and therefore $\pi_1 = 5/14$. Since $\pi_1 + \pi_2 = 1$ then $\pi_2 = 9/14$.