Queuing Example

Question

If we have a queue which has Poisson arrivals at rate λ and which serves customers (with a Poisson distributed service time) at a rate μ ($\lambda < \mu$) but which can hold only one customer at a time (an $M/M/1/1$ queue) then:

i) Write the differential-difference equations for $P_0(t)$ and $P_1(t)$ — the probability that the system is empty and has one customer.

ii) Solve the system to get $P_0(t)$ in terms of $P_0(0)$.

Use this to find the probability that a given arrival is turned away from the queue because it is busy serving a customer.

Answer

We get the following equations by considering transitions between the two states (remember the $o(\Delta t)$ term represents the possibility of a multiple event in a time period (anything other than a single birth, a single death or neither):

$$P_0(t + \Delta t) = (1 - \lambda\Delta t)P_0(t) + \mu P_1(t)\Delta t + o(\Delta t) \tag{1}$$

$$P_1(t + \Delta t) = (1 - \mu\Delta t)P_1(t) + \lambda P_0(t)\Delta t + o(\Delta t) \tag{2}$$

From the first:

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{o(\Delta t)}{\Delta t} \tag{3}$$

In the limit as $\Delta t \to 0$ this becomes:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \tag{4}$$

Now, since the only possibilities are these two states, clearly, $P_0(t) + P_1(t) = 1$. Therefore:

$$\frac{dP_0(t)}{dt} = -(\mu + \lambda)P_0(t) + \mu \tag{5}$$

By inspection the general solution is of the form:

$$P_0(t) = Ae^{-(\mu+\lambda)t} + B \tag{6}$$

where A and B are constants. Differentiating:

$$\frac{dP_0(t)}{dt} = -A(\mu + \lambda)e^{-(\mu+\lambda)t} \tag{7}$$

Combining with (5) gives:

$$-(\mu+\lambda)P_0(t)+\mu=-A(\mu+\lambda)e^{-(\mu+\lambda)t} \tag{8}$$

and therefore:

$$P_0(t)=Ae^{-(\mu+\lambda)t}+\frac{\mu}{\mu+\lambda} \tag{9}$$

Now, at $t=0$ we have:

$$P_0(0)=A+\frac{\mu}{\mu+\lambda} \tag{10}$$

Therefore $A=P_0(0)-\frac{\mu}{\mu+\lambda}$ giving us our final equation:

$$P_0(t)=\left[P_0(0)-\frac{\mu}{\mu+\lambda}\right]e^{-(\mu+\lambda)t}+\frac{\mu}{\mu+\lambda} \tag{11}$$

Note that, as $t\to\infty$ this will tend to $\mu/(\mu+\lambda)$.

We can perform similar calculations for $P_1(t)$ or simply use the fact that $P_0(t)+P_1(t)$ if we are smart to get that $P_1(t)\to\lambda/(\mu+\lambda)$ as $t\to\infty$. This answers the final question since it is the probability that an incoming customer will find itself blocked and rejected.