Lecture 3 — Extra Notes

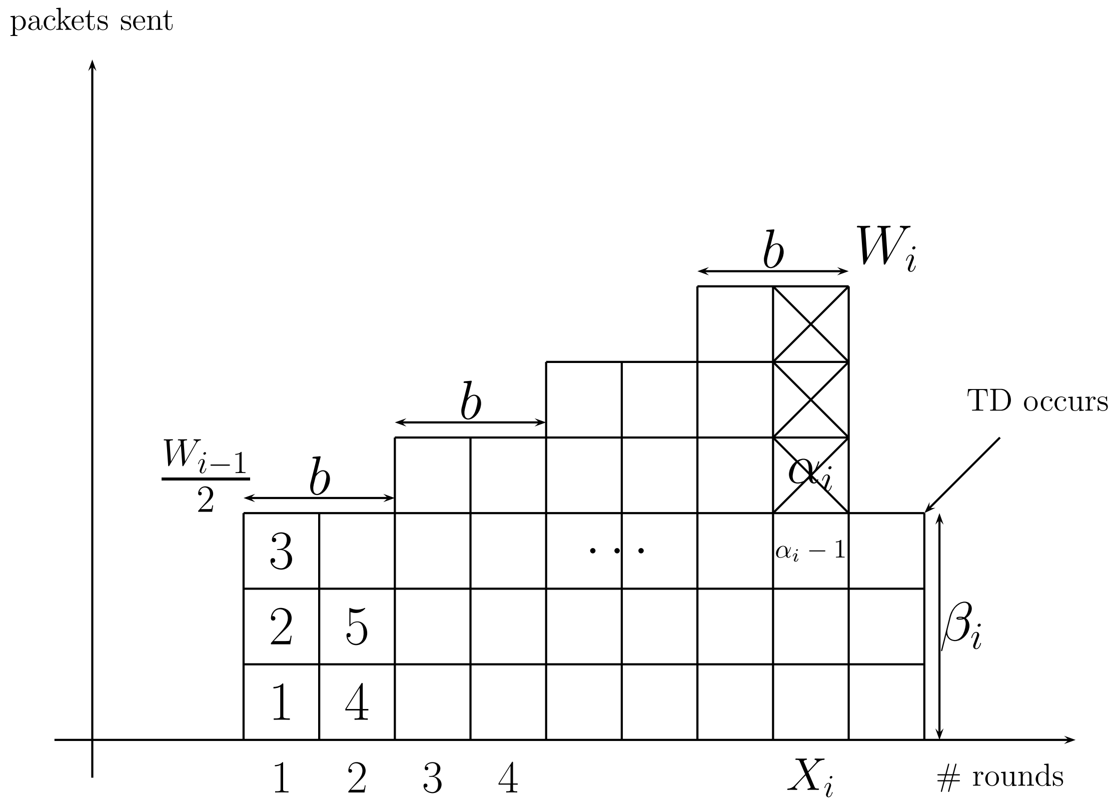

Figure 1: Diagram showing packet loss due to TD ACK

This work is taken from *Modelling TCP Throughput: A Simple Model and its Empirical Validation* by Padhye, Firoiu, Towsley and Kursoe.

Figure 1 shows in diagramtic form some of the quantities we need to define to study triple duplicate (TD) ACK packet loss.

Assume that there is a constant probability p of a packet being lost. We split our mathematical model into periods which occur between losses. W_i is the window size at the end of the ith period. Define Y_i as the number of packets send in the ith period and A_i as the time that the ith period took. It in the long term we can say that the bandwidth B is given by:

$$B = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{Y_i}{A_i} = \frac{E[Y]}{E[A]} \tag{1}$$

which gives the obvious idea that the bandwidth is expectation of the number of packets sent over the time taken to send them.

Consider figure 1. After a TD loss, the window size is halved (if we ignore the exponential *slow start* part of the algorithm). Therefore the ith period begins with a window size of $W_{i-1}/2$. If we split each period into *rounds* as shown in the diagram then, after every b successful rounds, the window size is incremented by one. Denote by α_i the number of the first packet lost in the

ith period (assume we number the packets from one beginning afresh every period). X_i is the number of the round in which this packet is lost.

Since, at the end of round i the window size is W_i then when this first packet is lost, $W_i - 1$ packets are outstanding already (since W_i is the number of unacknowledged packets at the end of the round). Therefore, a total of $Y_i = \alpha_i + W_i - 1$ packets are sent in the $X_i + 1$ rounds. Therefore:

$$E[Y] = E[\alpha] + E[W] - 1 \tag{2}$$

Now, if p is the probability that a packet is lost then the probability that the kth packet is the first lost is given by:

$$P[\alpha_i = k] = (1-p)^{k-1}p \quad k = 1, 2, \ldots \tag{3}$$

since this implies $k - 1$ were received then one was lost. Thus:

$$E[\alpha] = \sum_{k=1}^{\infty}(1-p)^{k-1}pk = \frac{1}{p} \tag{4}$$

A Brief Aside and Some Simple Probability Practice

It is instructive to go through this simple derivation since this is the type of manipulation we will encounter often in this course.

$$\sum_{k=1}^{\infty}(1-p)^{k-1}pk = p\sum_{k=1}^{\infty}q^{k-1}k$$

where $q = (1-p)$. We notice that:

$$p\sum_{k=1}^{\infty}q^{k-1}k = p\frac{d}{dq}\left(\sum_{k=1}^{\infty}q^k\right)$$

Now, if $0 < p < 1$ then we know that $\sum_{k=1}^{\infty} pq^{k-1} = 1$ since this is the sum of the probability over all states (that is the probability that packet loss ever occurs). Some trivial rearrangement gives us $\sum_{k=1}^{\infty} q^k = q/(1-q)$. Therefore substituting above:

$$\sum_{k=1}^{\infty}(1-p)^{k-1}pk = p\frac{d}{dq}\left(\frac{q}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Combining equations 2 and 4 we therefore have:

$$E[Y] = \frac{1-p}{p} + E[W] \tag{5}$$

Consider the round trip times of the packets (the time taken for a packet to be sent and an ACK to be received for it). The ith period has X_i complete rounds plus the partial round. We would therefore expect the duration of the round to be given by $A_i = (X_i + 1)RTT$ where RTT is the round trip time (assume that RTT is an independent random variable). Therefore:

$$E[A] = (E[X] + 1)RTT \tag{6}$$

Now, we need to know the value of $E[W]$. Since, every b rounds the window size increases by one and we have X_i rounds then clearly we have:

$$W_i = \frac{W_{i-1}}{2} + \frac{X_i}{b} \quad i = 1, 2, \ldots \tag{7}$$

[We should note that this is an approximation since it allows window sizes to be fractional when,obviously, they must really be integer]. The number of packets transmitted in the ith period is given by:

$$Y_i = \sum_{k=0}^{X_i/b-1} \left(\frac{W_{i-1}}{2} + k \right) b + \beta_i \tag{8}$$

Looking at the two terms in the sum, the first is a constant and the second is obtained simply from the formula for triangular numbers ($\sum_{i=0}^{n} i = n(n+1)/2$).

$$Y_i = \frac{X_i W_{i-1}}{2} + \frac{X_i}{2} \left(\frac{X_i}{b} - 1 \right) + \beta_i \tag{9}$$

Substitution from equation 7

$$Y_i = \frac{X_i}{2} \left(\frac{W_{i-1}}{2} + W_i - 1 \right) + \beta_i \tag{10}$$

From equation 7, if we make the assumption that X and W are i.i.d. random variables, (note that this assumption is not really necessary, we could model the behaviour of X_i as a Markov chain) we get:

$$E[W] = \frac{2}{b} E[X] \tag{11}$$

From equations 5 and 10 we get:

$$\frac{1-p}{p} + E[W] = \frac{E[X]}{2} \left(\frac{E[W]}{2} + E[W] - 1 \right) + E[\beta]] \tag{12}$$

Assuming that β_i is uniformly distributed between 1 and W_i we have $E[\beta] = W_i/2$. Combining 11 and 12 we get:

$$E[W] = \frac{2+b}{3b} + \sqrt{\frac{8(1-p)}{3bp} + \left(\frac{2+b}{3b} \right)^2} \tag{13}$$

Since p is likely to be small, it is worth noticing that:

$$E[W] = \sqrt{\frac{8}{3bp}} + o(1/\sqrt{p}) \tag{14}$$

Combining 13 and 11 we get:

$$E[X] = \frac{2+b}{6} + \sqrt{\frac{2b(1-p)}{3p} + \left(\frac{2+b}{6}\right)^2} \tag{15}$$

and combining this with 6 we have:

$$E[A] = RTT\left[\frac{2+b}{6} + \sqrt{\frac{2b(1-p)}{3p} + \left(\frac{2+b}{6}\right)^2} + 1\right] \tag{16}$$

And notice that:

$$E[X] = \sqrt{\frac{2b}{3p}} + o(1/\sqrt{p}) \tag{17}$$

From 1 and 2 we get:

$$B(p) = \frac{(1-p)/p + E[W]}{E[A]} = \frac{\frac{1-p}{p} + \frac{2+b}{6} + \sqrt{\frac{2b(1-p)}{3p} + \left(\frac{2+b}{6}\right)^2}}{RTT\left[\frac{2+b}{6} + \sqrt{\frac{2b(1-p)}{3p} + \left(\frac{2+b}{6}\right)^2} + 1\right]} \tag{18}$$

which we can express as:

$$B(p) = \frac{1}{RTT}\sqrt{\frac{3}{2bp}} + o(1/\sqrt{p}) \tag{19}$$