The mathematics of packet routing

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1 Reminder of the problem

The problem addressed in this lecture is the *routing problem*, that is the problem of finding out which links in the internet a packet must travel down to get from its origin to its destiniation. Remember that the internet is highly decentralised. It is not practical for any one router to keep a “map” of the whole internet.

You should always keep in mind that IPv4 (the current internet protocol) has no concept of route. That is to say, a packet of data does not specify its route and routers (unless they are running a special protocol) do not store the comnplete route from themselves to a desination but only the next “hop” that is on that route (that is only enough information to know where a packet should be sent next).

More information about the topics in this lecture can be found in [BG] (for a more mathematical viewpoint) and [Tan] (for an engineers look at networks).

1.1 Interesting aside – traceroute, a splendid hack

Some of you may be aware of the `traceroute` command. Many networks block it these days... if yours does then try `http://www.fifi.org/services/traceroute` to see it in action. Here is an example from an address in Estonia via an ISP in Rochdale to a home user.

```
traceroute to host213-121-67-224: (213.121.67.224): 2-20 hops, 38 byte packets
 2  213.180.11.162                    tondi-CR.online.ee    1.62 ms (ttl=127)
 3  213.180.25.1                      liiva-CR.online.ee    1.82 ms (ttl=126)
 4  213.180.11.189                      tix-CR.online.ee    2.16 ms (ttl=125)
 5  212.47.215.6     r1-Fa4-0-80-Tln-TIX.EE.KPNQwest.net    2.28 ms (ttl=251)
 6  134.222.224.5    r5-AT3-1.105.sthm-KPN1.SE.kpnqwest.net    12.2 ms (ttl=250)
 7  134.222.119.226  r2-Ge0-2-0-0.Sthm-KQ1.SE.KPNQwest.net    34.3 ms (ttl=246!)
 8  134.222.230.157  r2-Se0-3-0.hmbg-KQ2.DE.KPNQwest.net    33.4 ms (ttl=247!)
 9  134.222.230.117  r2-Se0-2-0.0.ffm-KQ1.DE.kpnqwest.net    34.1 ms (ttl=249!)
10  134.222.230.29   r2-Se0-3-0.0.ledn-KQ1.NL.kpnqwest.net    39.6 ms (ttl=248!)
11  134.222.230.169  r1-Se0-0-0.0.ldn-KQ1.UK.kpnqwest.net    43.7 ms (ttl=246!)
12  134.222.231.14   r1-Se0-0-0.0.Ldn-KQ4.UK.KPNQwest.net    44.9 ms (ttl=245!)
13  134.222.109.241  r13-Gi5-0.200.ldn-KQ4.UK.kpnqwest.net    45.4 ms (ttl=245!)
14  195.66.225.10                 linx-l1.ukcore.bt.net    45.2 ms (ttl=244!)
15  194.74.65.126     core2-pos14-0.ilford.ukcore.bt.net    45.3 ms (ttl=243!)
16  194.74.65.222     core2-pos5-0.reading.ukcore.bt.net    46.7 ms (ttl=242!)
17  62.6.196.109     core2-pos8-0.birmingham.ukcore.bt.net    54.3 ms (ttl=241!)
18  194.74.16.194    core2-pos9-0.rochdale.ukcore.bt.net    51.0 ms (ttl=240!)
19  217.32.168.5     vhsaccess1-gig1-0.rochdale.fixed.bt.net    51.1 ms (ttl=239!)
20  213.121.156.22   ugint0066-p.vhsaccess1.rochdale.fixed-nte.bt.net
                                                          51.3 ms (ttl=238!)
```

Traceroute appears to be retrieving a route – but how does it do this? In fact it isn't really getting a route. What it is doing is sending a series of packets with low TTL (time to live) to the same destination IP address. When the packets reach their expiry time then the receiving router sends back an ICMP "TTL exceeded" packet. This can be retrieved and the originating IP traced. A new packet is then sent with a TTL one higher. If your network adminstrator does not block ICMP then you can get an estimation of the route taken to the destination IP. Question: under what circumstances will this not be a connected route?

2 Basic graph theory

Definition 2.1. A graph $G = (\mathcal{N}, \mathcal{A})$ is a finite set of $\mathcal{N}$ nodes and a set $\mathcal{A}$ of *unordered* pairs (i, j) where $i, j \in: i \neq j$ (known as arcs). If n_1 and n_2 are nodes and (n_1, n_2) (where $n_1 \neq n_2$) is an arc then this arc is said to be *incident* on n_1 and n_2.

Definition 2.2. A directed graph $G = (\mathcal{N}, \mathcal{A})$ is a finite set of $\mathcal{N}$ nodes and a set $\mathcal{A}$ of *ordered* pairs (i, j) where $i, j \in \mathcal{N} : i \neq j$ (known as arcs).

Definition 2.3. A weighted graph is a graph (either directed or undirected) where every arc (i, j) has a weight w_{ij} associated with it.

These weights can be used in a variety of different ways. For example, they could represent the time taken for a packet to travel along a link, the delay the packet experiences, the level of congestion on the link, the cost a company would pay to use the link or even some combination of all of these.

Definition 2.4. A *walk* in a graph G is a sequence of nodes in a graph $(n_1, n_2, \ldots, n_l)$ such that each adjacent pair $(n_1, n_2), (n_2, n_3), \ldots (n_{l-1}, n_l)$ are arcs in G. A *path* is a walk with no repeated nodes. A *cycle* is a walk with $n_1 = n_l$ where $l > 3$ (for undirected graphs) or $l > 2$ (for directed graphs) and with no other repeated nodes. A *connected graph* is a graph where for any two nodes i and j we can find a walk which begins at i and ends at j.

Assume from here on that we are dealing with weighted, directed connected graphs.

With the definitions we have here we can formulate the routing problem as follows. The shortest route from $i \in \mathcal{N}$ to $j \in \mathcal{N}$ is the walk $n_1, n_2, \ldots, n_l, n_{l+1}$ with $n_1 = i$ and $n_{l+1} = j$ which minimises the sum $\sum_{i=1}^{l} w_{n_i n_{i+1}}$. In other words, the walk from i to j with the smallest weight over all arcs. It helps in formulating the problem to make assumption $w_{ij} = \infty$ if $(i,j) \notin \mathcal{N}$. Note that *shortest* here need not mean distance but could be any metric chosen depending on what we pick as weights.

3 Dijkstra's algorithm, a reminder

Dijkstra's Algorithm was discovered by the pioneering mathematician and programmer E W Dijkstra (1930 – 2002). The algorithm finds the shortest path to all nodes from an origin node, on a graph $G = (\mathcal{N}, \mathcal{A})$. It requires that all arc weights are non-negative – that is for all $(i,j) \in \mathcal{A} : w_{ij} \geq 0$.

Dijkstra's Algorithm involves the labelling of a set of permanent nodes P and node distances D_j to each node $j \in \mathcal{N}$. Assume that we wish to find the shortest paths from node 1 to all nodes. Then we begin with: $P = \{1\}$ and $D_1 = 0$ and $D_j = w_{1j}$ where $j \neq 1$. Dijkstra's algorithm then consists of following the procedure:

1. Find the next closest node. Find $i \notin P$ such that:
$$D_i = \min_{j \notin P} D_j$$
2. Update our set of permanently labelled nodes and our nodes distances: $P := P \cup \{i\}$.
3. If all nodes in $\mathcal{N}$ are also in P then we have finished so stop here.
4. Update the temporary (distance) labels for the new node i. For all $j \notin P$
$$D_j := \min[D_j, w_{ij} + D_i]$$
5. Go to the beginning of the algorithm.

Note that the version of this algorithm is subtly different from that in Bertsekas & Gallager which finds the paths from all nodes to a single destination. (This is not an important detail and it should be obvious how to reverse the algorithm). To understand the algorithm we make the following claims:

Proposition 3.1. *At the beginning of each iteration of Dijkstra's algorithm then:*

1. *$D_i \leq D_j$ for all $i \in P$ and $j \notin P$.*
2. *D_j is, for all j, the shortest distance from 1 to j using paths with all nodes (except, possibly j) in P.*

If this proposition can be proved then we can see that, when P contains every node in $\mathcal{N}$ then all the D_j are shortest paths by the second part of this proposition. Therefore proving the above proposition is equivalent to proving that Dijkstra's algorithm finds shortest paths.

The algorithm is proved in the appendices.

4 The Bellman–Ford algorithm

There is an obvious weakness with the Dijkstra algorithm as applied to the internet. It requires a knowledge of all the nodes links and costs to get the answer. By contrast the Bellman–Ford algorithm can be run in a distributed manner.

Notation. *D_i^h is the distance of the shortest walk from node 1 to node i of h steps or less.*

Set initially $D_i^0 = \infty$ for $i \neq 1$ and $D_1^h = 0$ for all h.

The Bellman-Ford Algorithm is then simply, for all $i \neq 1$,

$$D_i^{h+1} = \min_j [D_j^h + w_{ji}]$$

The algorithm terminates after h iterations if

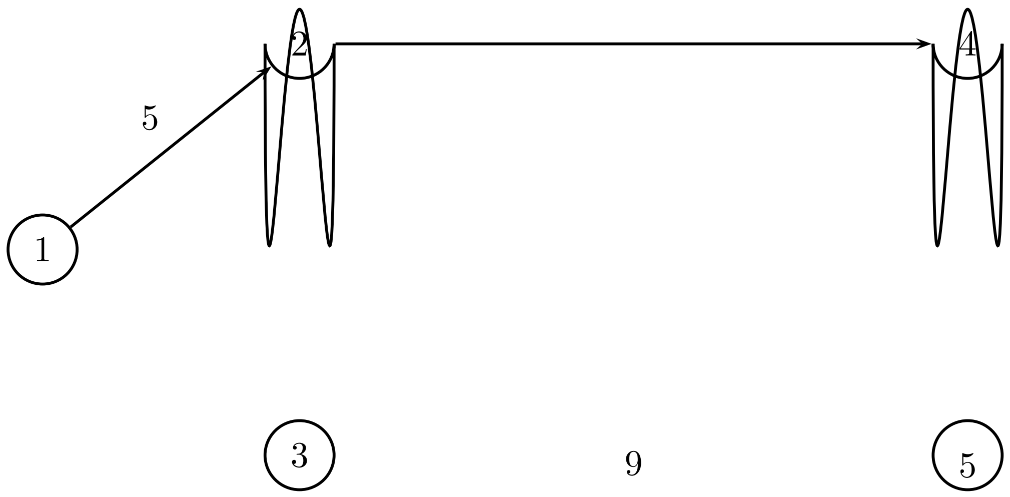
$$D_i^h = D_i^{h-1} \text{ for all } i$$

Proposition 4.1. *The scalars D_i^h generate by the algorithm from the starting values given for D_i^0 are equal to the shortest walk of length $\leq h$ from node 1 to node i.*

Proposition 4.2. *The algorithm terminates after a finite number of iterations if, and only if, all cycles not containing node i have non negative length. Furthermore, if the algorithm terminates, it does so after at most $h \leq N$ iterations and, at termination, D_i^h is the shortest path length from 1 to i.*

4.1 Bellman–Ford example

Here is an example network.



The algorithm then gives the answer.

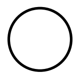
h	D_1^h	D_2^h	D_3^h	D_4^h	D_5^h
0	0	∞	∞	∞	∞
1	0	5	1	∞	∞
2	0	3	1	14	3
3	0	3	1	5	3
4	0	3	1	5	3

The good thing about the Bellman–Ford algorithm is that it can be run in a distributed manner. Imagine each node starts off knowing only the distance to its neighbours. At each time step the router reports to its neighbours the minimum distance it knows so far to each destination. In this way the algorithm is run without at any point any node having to build up a complete map of the network. The algorithm can also be performed in an asynchronous manner (that is with router all updating at different times rather than simultaneously). This is perfect for a network (like the internet) with no central clock.

4.2 Problems with routing

Routing with the Bellman–Ford algorithm is subject to a few problems. One of the best known is the *count to infinity* problem or *bad news phenomenon.*

Consider the network shown below where we want to find the route to node 1 – L is some large number.



Imagine the situation where Bellman–Ford has converged. Node 2 can get to node 1 with a cost of 3 (via 3 and 4). Node 3 can get to node 1 with a cost of 2 (via 4) and node 4 can get to node 1 directly.

Then the link $4 \to 1$ fails. All nodes should route via arc (2,1) but how many iterations of the Bellman–Ford algorithm does this take? Let D_n^h report the distance to node 1 reported at iteration n and the failure occurs at node n

h	D_1^h	D_2^h	D_3^h	D_4^h
n	0	3 (via 3)	2 (via 4)	1 (to 1)
$n+1$	0	3 (via 3)	2 (via 4)	4 (via 2)
$n+2$	0	3 (via 3)	5 (via 4)	4 (via 2)
$n+3$	0	6 (via 3)	5 (via 4)	4 (via 2)
$n+4$	0	6 (via 3)	5 (via 4)	7 (via 2)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n+L$	0	L (to 1)	$L+2$ (via 4)	$L+1$ (via 2)

References

[BG] Bersekas and Gallagher, *Data Networks (Second Edition)*

[Tan] Tanenbaum *Computer Networks (Third Edition)*

A Proof of Dijkstra's algorithm

Proposition A.1. *At the beginning of each iteration of Dijkstra's algorithm then:*

1. *$D_i \le D_j$ for all $i \in P$ and $j \notin P$.*
2. *D_j is, for all j, the shortest distance from 1 to j using paths with all nodes (except, possibly j) in P.*

If this proposition can be proved then we can see that, when P contains every node in $\mathcal{N}$ then all the D_j are shortest paths by the second part of this proposition. Therefore proving the above proposition is equivalent to proving that Dijkstra's algorithm finds shortest paths.

Proof. The proposition is trivially true at the first step since P consists only of the origin point (node 1) and D_j is 0 for $j = 1$, is $w_{ij} \ge 0$ for nodes reachable directly from node 1 and ∞ otherwise.

The first condition is simply shown to be satisfied since it is preserved by the formula:

$$D_j := \min[D_j, w_{ij} + D_i]$$

which is applied to all $j \notin P$ when node i is added to the set P.

We show the second condition by induction. We have established already that it is true at the very start of the algorithm. Let us assume it is true for the beginning of some iteration of the algorithm and show that it must then be true at the beginning of the next iteration.

Let node i be the node we are adding to our set P and let D_k be the label of each node k at the beginning of the step. The second condition must, therefore, hold for node $j = i$ (the new node we have added) by our induction hypothesis. It must also hold for all nodes $j \in P$ by part one of the proposition which is already proven. It remains to prove that the second condition of the proposition is met for $j \notin P \cup \{i\}$.

Consider a path from 1 to j which is shortest amongst those with all nodes except j in $P \cup \{i\}$ and let D'_j be the corresponding shortest distance. Such a path must contain a path from 1 to some node $r \in P \cup \{i\}$ and an arc(r, j). We have already established that the length of the path from 1 to r must be D_r and therefore we have:

$$D'_j = \min_{r \in P \cup \{i\}} [D_r + w_{rj}] = \min[\min_{r \in P}[D_r + w_{rj}], D_i + w_{ij}]$$

However, by our hypothesis $D_j = \min_{r \in P}[D_r + w_{rj}]$ therefore, $D'_j = min[D_j, D_i + w_{ij}]$ which is exactly what is set by the fourth step of the algorithm. Thus, after any iteration of the algorithm, the second part of the propostion is true if it was true at the beginning of the iteration. Thus the proof by induction is complete.

□

B Proof of the Bellman–Ford algorithm

Proposition B.1. *The scalars D_i^h generated by the algorithm from the starting values given for D_i^0 are equal to the shortest walk of length $\leq h$ from node 1 to node i.*

Proof. The first iteration will clearly give us $D_i^1 = w_{1i}$ for all i apart from $i = 1$ — which is, indeed, correct for the walk lengths of ≤ 1 from 1 to i. Let us suppose that D_i^k is the shortest walk of length $\leq k$ (from 1 to i) then complete the proof by induction by showing that D_i^{k+1} is the shortest walk of length $\leq k + 1$.

One possibility is that the shortest walk of length $\leq (k + 1)$ will be a walk of length k or less — in this case, $D_i^{k+1} = D_i^k$. Otherwise, the walk will be a walk of length $k + 1$ with the final arc being (j, i) added on to some walk of length k to node j. Thus, we can conclude that our shortest walk of length $k + 1$ is given by:

$$\text{Shortest walk to } i \text{ of length} \leq k + 1 = \min \left[D_i^k, \min_j D_j^k + w_{ji} \right]$$

However, since a walk of length $\leq k + 1$ must always be shorter or equal to a walk of length $\leq k$ (since the former contains the latter) then this reduces to

$$\text{Shortest walk to } i \text{ of length} \leq k + 1 = \min_j D_j^k + w_{ji}$$

which is our original expression for D_j^{k+1} and completes our proof by induction. □

Proposition B.2. *The algorithm terminates after a finite number of iterations if, and only if, all cycles not containing node i have non negative length. Furthermore, if the algorithm terminates, it does so after at most $h \leq N$ iterations and, at termination, D_i^h is the shortest path length from 1 to i.*

Proof. If negative length cycles exist then such cycles could be repeated as many times as desired to reduce the shortest walk length and thus the algorithm could never converge. Conversely, if no negative length cycles exist then any walks containing cycles could be made shorter or kept the same length by deleting such a cycle. Thus, our shortest walks contain no cycles. The maximum length of a walk with no cycles is $N-1$ (since such a walk will have covered every node). Thus, it trivially follows that $D_i^N = D_i^{N-1}$ and the algorithm terminates after, at most, N iterations. □